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LETTER TO THE EDITOR

Symmetry of vortex lattices in superfluid ^{3}He-A

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Abstract. Quantization of the circulation of superflow is considered for periodic vortex textures in the rotating A phase of superfluid ^{3}He. The quantization rule is derived by the application of symmetry principles alone.

The symmetry of the vortex lattice in superfluid ^{4}He [1] is quite simple: each elementary vortex-lattice cell contains an integer number of the singular vortices and, therefore, also an integer number n of the superflow circulation quanta. Consequently, the area A_{cell} of a primitive cell in the ^{4}He vortex lattice is determined by the following circulation condition:

$$\Omega A_{cell} = n\frac{h}{2m_4}. \tag{1}$$

This states that, on the average, the singular vorticity of the quantized vortex lines imitates solid-body rotation of the liquid.

An identical result has been derived by Volovik and Kopnin [2] *on the basis of topological arguments* for the contrasting case of an everywhere-continuous periodic vortex texture for the A phase in a rotating container:

$$\Omega A_{cell} = n\frac{h}{2m_3}. \tag{2}$$

Here we shall discuss the *symmetry of the vortex lattice* in superfluid ^{3}He and the consequent quantization rule for the area of the primitive lattice cell in the array formed by quantized vortex lines [3] in superfluid ^{3}He. In particular, we derive quite a general quantization rule for the area of a primitive lattice cell in any superfluid phase of liquid ^{3}He and for an arbitrary periodic vortex array structure with the help of symmetry principles.

Let us first find the symmetry group of the physical laws in an infinite sample of rotating liquid ^{3}He. This is readily found by considering only the flow of the normal fluid, which is determined by minimizing the following kinetic energy functional:

$$\int \mathrm{d}^3r\, \rho_n \left(v_n - \Omega \times r\right)^2. \tag{3}$$

This functional is not invariant under translations $t_a v_n = v_n(r - a)$ in the (x, y)-plane, but it does remain invariant under translations that are combined with the Galilean transformation Γ_u (defined through $\Gamma_u v_n \equiv v_n + u$). The combined transformation $\Gamma_{\Omega\times a} t_a$ keeps the energy functional in equation (3) invariant.

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Parity is conserved, $Pv_n = -v_n(-r)$, while time-inversion symmetry $Tv_n = -v_n$ is broken. Space rotations $SO_3^{(L)}$ reduce to rotations around $\hat{z}$; however, invariance prevails under the combined symmetry transformations constituted of a time inversion T and a rotation around x (and y) through the angle π. We denote the group containing these rotations and the combined rotations as $\mathcal{D}_\infty^{(LT)}$ because it is isomorphic with $\mathcal{D}_\infty$ (in the Landau–Lifshitz notation [4]).

All the results quoted above remain valid if one also includes the superfluid order parameter $A_{\alpha i}$ [5] which changes under Galilean transformations in the following fashion:

$$\Gamma_u A_{\alpha i} = e^{(2m_3/\hbar)u\cdot r} A_{\alpha i}. \tag{4}$$

Note that the combined symmetry $\Gamma_{\Omega\times a} t_a$ is equivalent to the symmetry introduced by Peierls [6] for a charged particle in a uniform magnetic field H. The latter is equivalent to rotation at the Larmor frequency $\Omega = (e/2mc)H$, where e and m denote the charge and mass of the particle.

Thus the total symmetry group $\mathcal{G}$ for superfluid ^{3}He in a rotating container is

$$\mathcal{G} = SO_3^{(S)} \times U(1) \times P \times \mathcal{D}_\infty^{(LT)} \times \Gamma t. \tag{5}$$

The enumeration of all the subgroups $\mathcal{H}$ of this group yields a classification of all the vortex states in a rotating container in terms of their symmetries $\mathcal{H}$.

For the simpler case of the possible states of an electron in a periodic lattice in the presence of a magnetic field, such a classification has been performed by Brown [7] and Zak [8]. In this case the electrons can only form periodic structures provided that the quantization condition on the magnetic flux is fulfilled: the total magnetic flux HA_{cell} through an elementary cell A_{cell} of the periodic lattice is quantized in units of the elementary flux quantum $\phi_0 = hc/e$:

$$HA_{cell} = n\phi_0. \tag{6}$$

An application of these ideas developed for electrons in a magnetic field to the case of superfluid ^{3}He in rotation yields the quantization condition for the area of the elementary lattice cell in the periodic vortex texture at a given angular velocity Ω in the rotating vessel. The periodic vortex lattices are described by the symmetry group $\mathcal{H}$, which contains combined translations Γt through the lattice periods. Let a_1 and a_2 denote the primitive lattice vectors; then $\mathcal{H}$ contains the elements

$$\Gamma_{\Omega\times a_1} t_{a_1} \tag{7}$$

and

$$\Gamma_{\Omega\times a_2} t_{a_2} \tag{8}$$

(or the above elements combined with gauge and/or spin rotations). Note that the order parameter $A_{\alpha i}$ is not periodic in the vortex lattice. The order parameter is only invariant under the combined translation

$$\Gamma_{\Omega\times a_1} t_{a_1} A_{\alpha i}(r) \equiv e^{(2m_3/\hbar)(\Omega\times a_1)\cdot r} A_{\alpha i}(r - a_1) = A_{\alpha i}(r). \tag{9}$$

Only those physical quantities that are Galilean invariant, such as $v_s - v_n$, are periodic in the vortex lattice.

In order to derive the quantization condition for the continuous vortex lattice, let us consider carrying out the transformations Γt along the closed contour of the boundary of the primitive cell in the periodic array. Since the result of these symmetry transformations does not change any physical quantities, including the order parameter, we find the following identity:

$$A_{\alpha i} = \Gamma_{\Omega\times(-a_2)} t_{(-a_2)} \Gamma_{\Omega\times(-a_1)} t_{(-a_1)} \Gamma_{\Omega\times a_2} t_{a_2} \Gamma_{\Omega\times a_1} t_{a_1} A_{\alpha i}. \tag{10}$$

Upon rearrangement of the noncommutative elements Γ and t we find the relationship

$$A_{\alpha i} = e^{i(4m_3/\hbar)\,\Omega\cdot(a_1\times a_2)}\, A_{\alpha i} \tag{11}$$

which yields us the quantization rule

$$\Omega \cdot A_{cell} = n\frac{h}{4m_3} \tag{12}$$

where $A_{cell} = a_1 \times a_2$ is the area of a primitive lattice cell in the periodic vortex-array structure. This is the analogue of the quantization condition (6) for the magnetic flux threading through the elementary cell in a periodic lattice in a superconductor.

For the equilibrium vortex texture lattice which imitates solid-body rotation such that

$$\langle \nabla \times v_s \rangle_{av} = 2\Omega \tag{13}$$

equation (11) finally yields the quantization condition for the circulation of superflow along the boundary of a lattice cell:

$$\oint_{cell\ boundary} \mathrm{d}r \cdot v_s = \int_{cell} \mathrm{d}A \cdot \nabla \times v_s = A_{cell} \cdot \langle \nabla \times v_s \rangle_{av} = 2\Omega \cdot A_{cell} = n\frac{h}{2m_3}. \tag{14}$$

On applying equation (14) to periodic arrangements of vortices in ^{3}He-A one can see from topological arguments that odd values of n always correspond to lattices of singular vortices, while among the structures with even values of n, there exist lattices of continuous vortex textures. For example, the lattice of continuous textures introduced by Volovik and Kopnin [2] has $n = 2$, while for the one suggested by Fujita, Nakahara, Ohmi and Tsuneto (see [9, 10]) one finds that $m = 4$.

While the symmetry of periodic A-phase vortex-lattice textures can thus be classified, the stability of the vortex lattices in a rotating superfluid remains beyond present considerations and is to be investigated in detail. See Baym [11] for an elegant recent discussion on the stability of the ^{4}He superfluid vortex lattice against thermal fluctuations associated with the long-wavelength Tkachenko modes.

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